Gear Geometry and Applied Theory

SECOND EDITION

Faydor L. Litvin
University of Illinois at Chicago

Alfonso Fuentes
Polytechnic University of Cartagena

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1 Coordinate Transformation

1.1 HOMOGENEOUS COORDINATES

A position vector in a three-dimensional space (Fig. 1.1.1) may be represented (i) in vector form as

$$\mathbf{r}_m = \overline{O_m M} = x_m \mathbf{i}_m + y_m \mathbf{j}_m + z_m \mathbf{k}_m \tag{1.1.1}$$

where ($\mathbf{i}_m$, $\mathbf{j}_m$, $\mathbf{k}_m$) are the unit vectors of coordinate axes, and (ii) by the column matrix

$$\mathbf{r}_m = \begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix}. \tag{1.1.2}$$

The subscript "m" indicates that the position vector is represented in coordinate system $S_m(x_m,\ y_m, z_m)$. To save space while designating a vector, we will also represent the position vector by the row matrix,

$$\mathbf{r}_m = [x_m \quad y_m \quad z_m]^{\mathrm{T}}. \tag{1.1.3}$$

The superscript "T" means that $\mathbf{r}_m^{\mathrm{T}}$ is a transpose matrix with respect to $\mathbf{r}_m$.

A point – the end of the position vector – is determined in Cartesian coordinates with three numbers: x, y, z. Generally, coordinate transformation in matrix operations needs mixed matrix operations where both multiplication and addition of matrices must be used. However, only multiplication of matrices is needed if position vectors are represented with homogeneous coordinates. Application of such coordinates for coordinate transformation in theory of mechanisms has been proposed by Denavit & Hartenberg [1955] and by Litvin [1955]. Homogeneous coordinates of a point in a three-dimensional space are determined by four numbers (x^*, y^*, z^*, t^*) which are not equal to zero simultaneously and of which only three are independent. Assuming that $t^* \neq 0$, ordinary coordinates and homogeneous coordinates may be related as follows:

$$x = \frac{x^*}{t^*} \quad y = \frac{y^*}{t^*} \quad z = \frac{z^*}{t^*}. \tag{1.1.4}$$

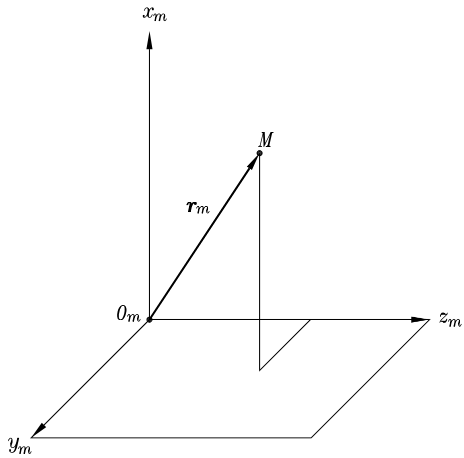

Figure 1.1.1: Position vector in Cartesian coordinate system.

With $t^* = 1$, a point may be specified by homogeneous coordinates such as $(x, y, z, 1)$, and a position vector may be represented by

$$\mathbf{r}_m = \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix} \text{ or}$$

$$\mathbf{r}_m = [x_m \quad y_m \quad z_m \quad 1]^{\mathrm{T}}.$$

1.2 COORDINATE TRANSFORMATION IN MATRIX REPRESENTATION

Consider two coordinate systems $S_m(x_m, y_m, z_m)$ and $S_n(x_n, y_n, z_n)$ (Fig. 1.2.1). Point M is represented in coordinate system S_m by the position vector

$$\mathbf{r}_m = [x_m \quad y_m \quad z_m \quad 1]^{\mathrm{T}}. \tag{1.2.1}$$

The same point M can be determined in coordinate system S_n by the position vector

$$\mathbf{r}_n = [x_n \quad y_n \quad z_n \quad 1]^{\mathrm{T}} \tag{1.2.2}$$

with the matrix equation

$$\mathbf{r}_n = \mathbf{M}_{nm}\mathbf{r}_m. \tag{1.2.3}$$

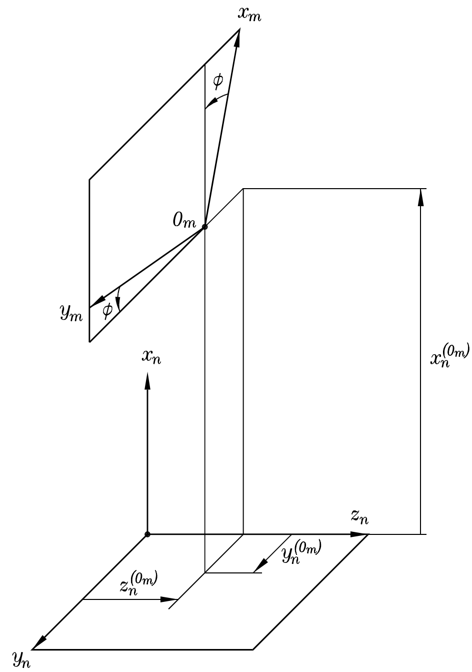

Figure 1.2.1: Derivation of coordinate transformation.

Matrix $\mathbf{M}_{nm}$ is represented by

$$\mathbf{M}_{nm} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{i}_n \cdot \mathbf{i}_m) & (\mathbf{i}_n \cdot \mathbf{j}_m) & (\mathbf{i}_n \cdot \mathbf{k}_m) & (\overline{O_n O_m} \cdot \mathbf{i}_n) \\ (\mathbf{j}_n \cdot \mathbf{i}_m) & (\mathbf{j}_n \cdot \mathbf{j}_m) & (\mathbf{j}_n \cdot \mathbf{k}_m) & (\overline{O_n O_m} \cdot \mathbf{j}_n) \\ (\mathbf{k}_n \cdot \mathbf{i}_m) & (\mathbf{k}_n \cdot \mathbf{j}_m) & (\mathbf{k}_n \cdot \mathbf{k}_m) & (\overline{O_n O_m} \cdot \mathbf{k}_n) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\widehat{x_n, x_m}) & \cos(\widehat{x_n, y_m}) & \cos(\widehat{x_n, z_m}) & x_n^{(O_m)} \\ \cos(\widehat{y_n, x_m}) & \cos(\widehat{y_n, y_m}) & \cos(\widehat{y_n, z_m}) & y_n^{(O_m)} \\ \cos(\widehat{z_n, x_m}) & \cos(\widehat{z_n, y_m}) & \cos(\widehat{z_n, z_m}) & z_n^{(O_m)} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (1.2.4)$$

Here, $(\mathbf{i}_n, \mathbf{j}_n, \mathbf{k}_n)$ are the unit vectors of the axes of the "new" coordinate system; $(\mathbf{i}_m, \mathbf{j}_m, \mathbf{k}_m)$ are the unit vectors of the axes of the "old" coordinate system; O_n and O_m are the origins of the "new" and "old" coordinate systems; subscript "nm" in the designation $\mathbf{M}_{nm}$ indicates that the coordinate transformation is performed from S_m to

S_n. The determination of elements a_{lk} $(k = 1, 2, 3; l = 1, 2, 3)$ of matrix $\mathbf{M}_{nm}$ is based on the following rules:

(i) Elements of the 3×3 submatrix

$$\mathbf{L}_{nm} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1.2.5}$$

represent the direction cosines of the "old" unit vectors $(\mathbf{i}_m, \mathbf{j}_m, \mathbf{k}_m)$ in the "new" coordinate systems S_n. For instance, $a_{21} = \cos(\widehat{y_n, x_m})$, $a_{32} = \cos(\widehat{z_n, y_m})$, and so on. The subscripts of elements a_{kl} in matrix (1.2.5) indicate the number l of the "old" coordinate axis and the number k of the "new" coordinate axis. Axes x, y, z are given numbers 1, 2, and 3, respectively.

(ii) Elements $a_{14},\ a_{24}$, and a_{34} represent the "new" coordinates $x_n^{(O_m)}$, $y_n^{(O_m)}$, $z_n^{(O_m)}$ of the "old" origin O_m.

Recall that nine elements of matrix $\mathbf{L}_{nm}$ are related by six equations that express the following:

(1) Elements of each row (or column) are direction cosines of a unit vector. Thus,

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1, \qquad a_{11}^2 + a_{21}^2 + a_{31}^2 = 1, \qquad \cdots . \tag{1.2.6}$$

(2) Due to orthogonality of unit vectors of coordinate axes, we have

$$\begin{aligned} &[a_{11}\ a_{12}\ a_{13}]\,[a_{21}\ a_{22}\ a_{23}]^{\mathrm{T}} = 0 \\ &[a_{11}\ a_{21}\ a_{31}]\,[a_{12}\ a_{22}\ a_{32}]^{\mathrm{T}} = 0. \end{aligned} \tag{1.2.7}$$

An element of matrix $\mathbf{L}_{nm}$ can be represented by a respective determinant of the second order [Strang, 1988]. For instance,

$$a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad a_{23} = (-1) \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}. \tag{1.2.8}$$

To determine the new coordinates $(x_n, y_n, z_n, 1)$ of point M, we have to use the rule of multiplication of a square matrix (4×4) and a column matrix (4×1). (*The number of rows in the column matrix is equal to the number of columns in matrix* $\mathbf{M}_{nm}$.) Equation (1.2.3) yields

$$\begin{aligned} x_n &= a_{11}x_m + a_{12}y_m + a_{13}z_m + a_{14} \\ y_n &= a_{21}x_m + a_{22}y_m + a_{23}z_m + a_{24} \\ z_n &= a_{31}x_m + a_{32}y_m + a_{33}z_m + a_{34}. \end{aligned} \tag{1.2.9}$$

The purpose of the *inverse* coordinate transformation is to determine the coordinates (x_m, y_m, z_m), taking as given coordinates (x_n, y_n, z_n). The inverse coordinate transformation is represented by

$$\mathbf{r}_m = \mathbf{M}_{mn}\mathbf{r}_n. \tag{1.2.10}$$

The inverse matrix $\mathbf{M}_{mn}$ indeed exists if the determinant of matrix $\mathbf{M}_{nm}$ differs from zero.

There is a simple rule that allows the elements of the inverse matrix to be determined in terms of elements of the direct matrix. Consider that matrix $\mathbf{M}_{nm}$ is given by

$$\mathbf{M}_{nm} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.2.11}$$

It is necessary to determine the elements of matrix $\mathbf{M}_{mn}$ represented by

$$\mathbf{M}_{mn} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.2.12}$$

Here,

$$\mathbf{M}_{mn} = \mathbf{M}_{nm}^{-1}, \quad \mathbf{M}_{mn}\mathbf{M}_{nm} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix.

The submatrix $\mathbf{L}_{mn}$ of the order (3×3) is determined as follows:

$$\mathbf{L}_{mn} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \mathbf{L}_{nm}^{\mathrm{T}}. \tag{1.2.13}$$

The remaining elements (b_{14}, b_{24}, and b_{34}) are determined with the following equations:

$$b_{14} = -(a_{11}a_{14} + a_{21}a_{24} + a_{31}a_{34}) \Rightarrow - \begin{bmatrix} :a_{11}: & a_{12} & a_{13} & :a_{14}: \\ :a_{21}: & a_{22} & a_{23} & :a_{24}: \\ :a_{31}: & a_{32} & a_{33} & :a_{34}: \\ :0: & 0 & 0 & :1: \end{bmatrix}$$

$$b_{24} = -(a_{12}a_{14} + a_{22}a_{24} + a_{32}a_{34}) \Rightarrow - \begin{bmatrix} a_{11} & :a_{12}: & a_{13} & :a_{14}: \\ a_{21} & :a_{22}: & a_{23} & :a_{24}: \\ a_{31} & :a_{32}: & a_{33} & :a_{34}: \\ 0 & :0: & 0 & :1: \end{bmatrix}$$

$$b_{34} = -(a_{13}a_{14} + a_{23}a_{24} + a_{33}a_{34}) \Rightarrow - \begin{bmatrix} a_{11} & a_{12} & :a_{13}: & :a_{14}: \\ a_{21} & a_{22} & :a_{23}: & :a_{24}: \\ a_{31} & a_{32} & :a_{33}: & :a_{34}: \\ 0 & 0 & :0: & :1: \end{bmatrix}. \tag{1.2.14}$$

The columns to be multiplied are marked.

To perform successive coordinate transformation, we need only to follow the product rule of matrix algebra. For instance, the matrix equation

$$\mathbf{r}_p = \mathbf{M}_{p(p-1)}\mathbf{M}_{(p-1)(p-2)} \cdots \mathbf{M}_{32}\mathbf{M}_{21}\mathbf{r}_1 \tag{1.2.15}$$

represents successive coordinate transformation from S_1 to S_2, from S_2 to $S_3, \ldots,$ from S_{p-1} to S_p.

To perform transformation of components of free vectors, we need only to apply 3×3 submatrices **L**, which may be obtained by eliminating the last row and the last column of the corresponding matrix **M**. This results from the fact that the free-vector components (projections on coordinate axes) do not depend on the location of the origin of the coordinate system.

The transformation of vector components of a free vector **A** from system S_m to S_n is represented by the matrix equation

$$\mathbf{A}_n = \mathbf{L}_{nm}\mathbf{A}_m \tag{1.2.16}$$

where

$$\mathbf{A}_n = \begin{bmatrix} A_{xn} \\ A_{yn} \\ A_{zn} \end{bmatrix}, \quad \mathbf{L}_{nm} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}_m = \begin{bmatrix} A_{xm} \\ A_{ym} \\ A_{zm} \end{bmatrix}. \tag{1.2.17}$$

A normal to the gear tooth surface is a *sliding* vector because it may be translated along its line of action. However, we may transform the surface normal as a *free* vector if the surface point where the surface normal is considered will be transferred simultaneously.

1.3 ROTATION ABOUT AN AXIS

Two Main Problems

We consider a general case in which the rotation is performed about an axis that does not coincide with any axis of the employed coordinate system. We designate the unit vector of the axis of rotation by **c** (Fig. 1.3.1) and assume that the rotation about **c** may be performed either counterclockwise or clockwise.

Henceforth we consider two coordinate systems: (i) the fixed one, S_a; and (ii) the movable one, S_b. There are two typical problems related to rotation about **c**. The first one can be formulated as follows.

Consider that a position vector is rigidly connected to the movable body. The initial position of the position vector is designated by $\overline{OA} = \boldsymbol{\rho}$ (Fig. 1.3.1). After rotation through an angle ϕ about **c**, vector $\boldsymbol{\rho}$ will take a new position designated by $\overline{OA}^* = \boldsymbol{\rho}^*$. Both vectors, $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^*$ (Fig. 1.3.1), are considered to be in the *same* coordinate system, say S_a. Our goal is to develop an equation that relates components of vectors $\boldsymbol{\rho}_a$ and $\boldsymbol{\rho}_a^*$. (The subscript "a" indicates that the two vectors are represented in the *same* coordinate system S_a.) Matrix equation

$$\boldsymbol{\rho}_a^* = \mathbf{L}_a\boldsymbol{\rho}_a \tag{1.3.1}$$

describes the relation between the components of vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^*$ that are represented in the same coordinate system S_a.

The other problem concerns representation of the *same* position vector in different coordinate systems. Our goal is to derive matrix $\mathbf{L}_{ba}$ in matrix equation

$$\boldsymbol{\rho}_b = \mathbf{L}_{ba}\boldsymbol{\rho}_a. \tag{1.3.2}$$

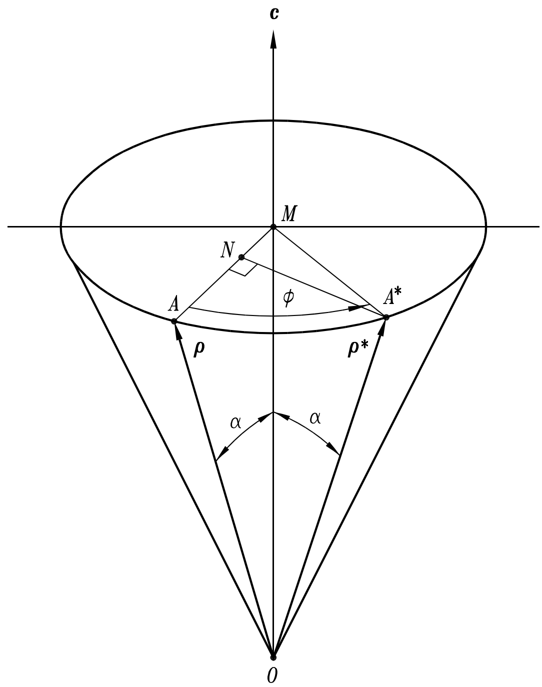

Figure 1.3.1: Rigid body rotation.

The designations $\boldsymbol{\rho}_a$ and $\boldsymbol{\rho}_b$ indicate that the *same* position vector $\boldsymbol{\rho}$ is represented in coordinate systems S_a and S_b, respectively. Although the *same* position vector is considered, the components of $\boldsymbol{\rho}$ in coordinate systems S_a and S_b are different and we designate them by

$$\boldsymbol{\rho}_a = a_1\mathbf{i}_a + a_2\mathbf{j}_a + a_3\mathbf{k}_a \tag{1.3.3}$$

and

$$\boldsymbol{\rho}_b = b_1\mathbf{i}_b + b_2\mathbf{j}_b + b_3\mathbf{k}_b. \tag{1.3.4}$$

Matrix $\mathbf{L}_{ba}$ is an operator that transforms the components $[a_1 \quad a_2 \quad a_3]^{\mathrm{T}}$ into $[b_1 \quad b_2 \quad b_3]^{\mathrm{T}}$. It will be shown below that operators $\mathbf{L}_a$ and $\mathbf{L}_{ba}$ are related.

Problem 1. Relations between components of vectors $\boldsymbol{\rho}_a$ and $\boldsymbol{\rho}_a^*$.

Recall that $\boldsymbol{\rho}_a$ and $\boldsymbol{\rho}_a^*$ are two position vectors that are represented in the *same* coordinate system S_a. Vector $\boldsymbol{\rho}$ represents the initial position of the position vector, *before* rotation, and $\boldsymbol{\rho}^*$ represents the position vector *after* rotation about $\mathbf{c}$. The following derivations are based on the assumption that rotation about $\mathbf{c}$ is performed counterclockwise. The procedure of derivations (see also Suh & Radcliffe, 1978, Shabana, 1989, and others) is as follows.

Step 1: We represent $\boldsymbol{\rho}_a^*$ by the equation (Fig. 1.3.1)

$$\boldsymbol{\rho}_a^* = \overline{OM} + \overline{MN} + \overline{NA^*} \tag{1.3.5}$$

where

$$\overline{OM} = (\mathbf{c}_a \cdot \boldsymbol{\rho}_a)\mathbf{c}_a = (\mathbf{c}_a \cdot \boldsymbol{\rho}_a^*)\mathbf{c}_a \tag{1.3.6}$$

and $\mathbf{c}_a$ is the unit vector of the axis of rotation that is represented in S_a.

Step 2: Vector $\boldsymbol{\rho}_a$ is represented by the equation

$$\boldsymbol{\rho}_a = \overline{OM} + \overline{MA} = (\mathbf{c}_a \cdot \boldsymbol{\rho}_a)\mathbf{c}_a + \overline{MA} \tag{1.3.7}$$

that yields

$$\overline{MA} = \boldsymbol{\rho}_a - (\mathbf{c}_a \cdot \boldsymbol{\rho}_a)\mathbf{c}_a. \tag{1.3.8}$$

We emphasize that a vector being rotated about $\mathbf{c}$ generates a cone with an apex angle α. Thus, both vectors, $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^*$, are the generatrices of the same cone, as shown in Fig. 1.3.1.

Step 3: Vector $\overline{MN}$ has the same direction as $\overline{MA}$ and this yields

$$|\overline{MN}| = |\overline{MA^*}| \cos\phi = |\overline{MA}| \cos\phi = \rho \sin\alpha \cos\phi \tag{1.3.9}$$

where α is the apex angle of the generated cone, $|\overline{MA}| = \rho \sin\alpha$, and ρ is the magnitude of $\boldsymbol{\rho}$.

Equations (1.3.8) and (1.3.9) yield

$$\overline{MN} = |\overline{MN}| \frac{\overline{MA}}{|\overline{MA}|} = [\boldsymbol{\rho}_a - (\mathbf{c}_a \cdot \boldsymbol{\rho}_a)\mathbf{c}_a] \cos\phi. \tag{1.3.10}$$

Step 4: Vector $\overline{NA^*}$ has the same direction as $(\mathbf{c}_a \times \boldsymbol{\rho}_a)$ and may be represented by

$$\overline{NA^*} = \frac{\mathbf{c}_a \times \boldsymbol{\rho}_a}{|\mathbf{c}_a \times \boldsymbol{\rho}_a|} |\overline{NA^*}| = \sin\phi(\mathbf{c}_a \times \boldsymbol{\rho}_a). \tag{1.3.11}$$

Here,

$$|\overline{NA^*}| = |\overline{MA^*}| \sin\phi = \rho \sin\alpha \sin\phi, \quad |\mathbf{c}_a \times \boldsymbol{\rho}_a| = \rho \sin\alpha.$$

Step 5: Equations (1.3.5), (1.3.6), (1.3.10), and (1.3.11) yield

$$\boldsymbol{\rho}_a^* = \boldsymbol{\rho}_a \cos\phi + (1 - \cos\phi)(\mathbf{c}_a \cdot \boldsymbol{\rho}_a)\mathbf{c}_a + \sin\phi(\mathbf{c}_a \times \boldsymbol{\rho}_a). \tag{1.3.12}$$

Step 6: It is easy to prove that

$$(\mathbf{c}_a \cdot \boldsymbol{\rho}_a)\mathbf{c}_a = \mathbf{c}_a \times (\mathbf{c}_a \times \boldsymbol{\rho}_a) + \boldsymbol{\rho}_a \tag{1.3.13}$$

because

$$\mathbf{c}_a \times (\mathbf{c}_a \times \boldsymbol{\rho}_a) = (\mathbf{c}_a \cdot \boldsymbol{\rho}_a)\mathbf{c}_a - \boldsymbol{\rho}_a(\mathbf{c}_a \cdot \mathbf{c}_a).$$

Step 7: Equations (1.3.12) and (1.3.13) yield

$$\boldsymbol{\rho}_a^* = \boldsymbol{\rho}_a + (1 - \cos\phi)[\mathbf{c}_a \times (\mathbf{c}_a \times \boldsymbol{\rho}_a)] + \sin\phi(\mathbf{c}_a \times \boldsymbol{\rho}_a). \tag{1.3.14}$$

Equation (1.3.14) is known as the Rodrigues formula. According to the investigation by Cheng & Gupta [1989], this equation deserves to be called the Euler–Rodrigues, formula.

Step 8: Additional derivations are directed at representation of the Euler–Rodrigues formula in matrix form.

The cross product ($\mathbf{c}_a \times \boldsymbol{\rho}_a$) may be represented in matrix form by

$$\mathbf{c}_a \times \boldsymbol{\rho}_a = \mathbf{C}^s \boldsymbol{\rho}_a \tag{1.3.15}$$

where $\mathbf{C}^s$ is the skew-symmetric matrix represented by

$$\mathbf{C}^s = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}. \tag{1.3.16}$$

Vector $\mathbf{c}_a$ is represented by

$$\mathbf{c}_a = c_1 \mathbf{i}_a + c_2 \mathbf{j}_a + c_3 \mathbf{k}_a. \tag{1.3.17}$$

Step 9: Equations (1.3.14), (1.3.15), and (1.3.16) yield the following matrix representation of the Euler–Rodrigues formula:

$$\boldsymbol{\rho}_a^* = \left[\mathbf{I} + (1 - \cos\phi)(\mathbf{C}^s)^2 + \sin\phi \mathbf{C}^s\right] \boldsymbol{\rho}_a = \mathbf{L}_a \boldsymbol{\rho}_a \tag{1.3.18}$$

where $\mathbf{I}$ is the 3×3 identity matrix. While deriving Eqs. (1.3.14) and (1.3.18), we assumed that the rotation is performed counterclockwise. For the case of clockwise rotation, it is necessary to change the sign preceding $\sin\phi$ to its opposite. The expression for matrix $\mathbf{L}_a$ that will cover two directions of rotation is

$$\mathbf{L}_a = \mathbf{I} + (1 - \cos\phi)(\mathbf{C}^s)^2 \pm \sin\phi \mathbf{C}^s. \tag{1.3.19}$$

The upper sign preceding $\sin\phi$ corresponds to counterclockwise rotation and the lower sign corresponds to rotation in a clockwise direction. In both cases the unit vector $\mathbf{c}$ must be expressed by the same Eq. (1.3.17) that determines the orientation of $\mathbf{c}$ but not the direction of rotation. The direction of rotation is identified with the proper sign preceding $\sin\phi$ in Eq. (1.3.19).

Problem 2. Recall that our goal is to derive the operator $\mathbf{L}_{ba}$ in matrix equation (1.3.2) that transforms components of the *same* vector (see Eqs. (1.3.3) and (1.3.4)). It will be shown below that the sought-for operator is represented as

$$\mathbf{L}_{ba} = \mathbf{L}_a^{\mathrm{T}} = \mathbf{I} + (1 - \cos\phi)(\mathbf{C}^s)^2 \mp \sin\phi \mathbf{C}^s. \tag{1.3.20}$$

Operator $\mathbf{L}_{ba}$ can be obtained from operator $\mathbf{L}_a$ given by Eq. (1.3.19) by changing the sign of the angle of rotation, ϕ. The upper and lower signs preceding $\sin\phi$ in Eq. (1.3.20) correspond to the cases where S_a will coincide with S_b by rotation counterclockwise and clockwise, respectively. The proof is based on the determination of components of the same vector, say vector $\overline{OA}$ shown in Fig. 1.3.1, in coordinate systems S_a and S_b.

Step 1: We consider initially that vector $\overline{OA}$ is represented in S_a as

$$\boldsymbol{\rho}_a = [a_1 \;\; a_2 \;\; a_3]^{\mathrm{T}}. \tag{1.3.21}$$

Step 2: To determine components of vector $\overline{OA}$ in S_b we consider first that coordinate system S_b and the previously mentioned position vector are rotated as one rigid body about $\mathbf{c}$. After rotation through angle ϕ, position vector $\overline{OA}$ will take the position $\overline{OA}^*$ and can be represented in S_b as

$$\overline{OA}^* = a_1\mathbf{i}_b + a_2\mathbf{j}_b + a_3\mathbf{k}_b. \tag{1.3.22}$$

It is obvious that vector $\overline{OA}^*$ has in S_b the same components as vector $\overline{OA}$ has in S_a.

Step 3: We consider now in S_b two vectors $\overline{OA}^*$ and $\overline{OA}$. Vector $\overline{OA}^*$ will coincide with $\overline{OA}$ after clockwise rotation about $\mathbf{c}$. The components of vectors $\overline{OA}^*$ and $\overline{OA}$ in S_b are related by an equation that is similar to Eq. (1.3.19). The difference is that we now have to consider that the rotation from $\overline{OA}^*$ to $\overline{OA}$ is performed clockwise. Then we obtain

$$(\overline{OA})_b = \mathbf{L}_b(\overline{OA}^*)_b = \left[\mathbf{I} + (1-\cos\phi)(\mathbf{C}^s)^2 - \sin\phi\mathbf{C}^s\right](\overline{OA}^*)_b. \tag{1.3.23}$$

Designating components of $(\overline{OA})_b$ by $[b_1 \;\; b_2 \;\; b_3]^{\mathrm{T}}$, we receive

$$[b_1 \;\; b_2 \;\; b_3]^{\mathrm{T}} = [\mathbf{I} + (1-\cos\phi)(\mathbf{C}^s)^2 - \sin\phi\mathbf{C}^s][a_1 \;\; a_2 \;\; a_3]^{\mathrm{T}}. \tag{1.3.24}$$

Step 4: We have now obtained components of the same vector $\overline{OA}$ in coordinate systems S_a and S_b, respectively. The matrix equation that describes transformation of components of $\overline{OA}$ is

$$(\overline{OA})_b = \mathbf{L}_{ba}(\overline{OA})_a. \tag{1.3.25}$$

For the case in which rotation from S_a to S_b is performed counterclockwise we have obtained that

$$\mathbf{L}_{ba} = \mathbf{I} + (1-\cos\phi)(\mathbf{C}^s)^2 - \sin\phi\mathbf{C}^s. \tag{1.3.26}$$

Similarly, for the case in which rotation from S_a to S_b is performed clockwise, we obtain

$$\mathbf{L}_{ba} = \mathbf{I} + (1-\cos\phi)(\mathbf{C}^s)^2 + \sin\phi\mathbf{C}^s. \tag{1.3.27}$$

The general description of operator $\mathbf{L}_{ba}$ and the respective coordinate transformation are as follows:

$$\boldsymbol{\rho}_b = \mathbf{L}_{ba}\boldsymbol{\rho}_a = \left[\mathbf{I} + (1-\cos\phi)(\mathbf{C}^s)^2 \mp \sin\phi\mathbf{C}^s\right]\boldsymbol{\rho}_a. \tag{1.3.28}$$

The upper and lower signs preceding $\sin\phi$ correspond to the cases in which rotation from S_a to S_b is performed counterclockwise and clockwise, respectively.

In our identification of coordinate systems S_a and S_b we do not use the terms fixed and movable. We just consider that S_a is the previous coordinate system and S_b is the new one, and we take into account how the rotation from S_a to S_b is performed: counterclockwise or clockwise.

Matrix L_{ba}

Using Eqs. (1.3.26) and (1.3.27), we may represent elements of matrix $\mathbf{L}_{ba}$ in terms of components of unit vector $\mathbf{c}$ of the axis of rotation and the angle of rotation ϕ. Thus,

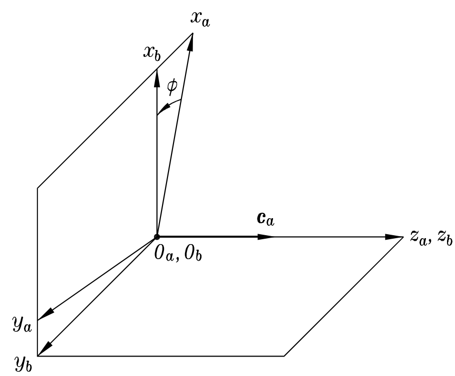

Figure 1.3.2: Derivation of coordinate transformation by rotation.

we obtain

$$\mathbf{L}_{ba} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{1.3.29}$$

Here,

$$\begin{aligned} a_{11} &= \cos\phi\left(1 - c_1^2\right) + c_1^2 \\ a_{12} &= (1 - \cos\phi)c_1c_2 \pm \sin\phi c_3 \\ a_{13} &= (1 - \cos\phi)c_1c_3 \mp \sin\phi c_2 \\ a_{21} &= (1 - \cos\phi)c_1c_2 \mp \sin\phi c_3 \\ a_{22} &= \cos\phi\left(1 - c_2^2\right) + c_2^2 \\ a_{23} &= (1 - \cos\phi)c_2c_3 \pm \sin\phi c_1 \\ a_{31} &= (1 - \cos\phi)c_1c_3 \pm \sin\phi c_2 \\ a_{32} &= (1 - \cos\phi)c_2c_3 \mp \sin\phi c_1 \\ a_{33} &= \cos\phi\left(1 - c_3^2\right) + c_3^2. \end{aligned} \tag{1.3.30}$$

When the axis of rotation coincides with a coordinate axis of S_a, we have to make two components of unit vector $\mathbf{c}_a$ equal to zero in Eqs. (1.3.30). For instance, in the case in which rotation is performed about the z_a axis (Fig. 1.3.2), we have

$$\mathbf{c}_a = \mathbf{k}_a = [0 \ \ 0 \ \ 1]^{\mathrm{T}}. \tag{1.3.31}$$

We emphasize again that in all cases of coordinate transformation only elements (1.3.30) of matrix $\mathbf{L}_{ba}$, and not the components of $\mathbf{c}_a$, depend on the direction of rotation. The unit vector $\mathbf{c}$ can be represented in either of the two coordinate systems, S_a and S_b, by the equations

$$\mathbf{c} = c_1\mathbf{i}_a + c_2\mathbf{j}_a + c_3\mathbf{k}_a = c_1\mathbf{i}_b + c_2\mathbf{j}_b + c_3\mathbf{k}_b. \tag{1.3.32}$$

This means that the unit vector $\mathbf{c}$ of the axis of rotation has the *same* components in both coordinate systems, S_a and S_b. It is easily verified that

$$[c_1 \ \ c_2 \ \ c_3]^{\mathrm{T}} = \mathbf{L}_{ba}[c_1 \ \ c_2 \ \ c_3]^{\mathrm{T}}. \tag{1.3.33}$$